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Advanced Microeconomics

Enci Liu el3334@Columbia.edu
Beijing, CN

1. Preference and Choice

There are two distinct approaches to modeling individual choice behavior.

The first one treats the decision makers' tastes, as summarized in their **Preference Relation**, as the primitive characteristic of the individual.

The second approach treats the individual's choice behavior as the primitive feature and proceeds by making assumptions directly concerning this behavior. A central assumption in this approach is the **Weak Axiom of Revealed Preference**.

1.1. Preference Relations

In the preference-based approach, the objectives of the decision maker are summarized in a preference relation, which denote by $\succeq$. $\succeq$ is a binary relation on the set of alternatives X, allowing the comparison of pairs of alternatives $x, y \in X$.

Definition 1

(1) The **Prefence Relation** $\succeq$ is defined as:

$$x \succeq y \Leftrightarrow x \text{ is at least as good as } y$$

(2) The **Strict Prefence Relation** $\succ$ is defined as:

$$x \succ y \Leftrightarrow x \succeq y \wedge y \not\succeq x$$

(3) The **Indifference Relation** $\sim$ is defined as:

$$x \sim y \Leftrightarrow x \succeq y \wedge y \succeq x$$

Definition 2 The preference relation $\succeq$ is **Rational** if it is

(1) Complete: $\forall x, y \in X, x \succeq y \vee y \succeq x$.

(2) Transitive: $\forall x, y, z \in X, x \succeq y \wedge y \succeq z \Rightarrow x \succeq z$.

Theorem 3 If $\succeq$ is rational, then

(1) $\succ$ is irreflective ($\forall x \in X, x \not\succ x$) and transitive.

(2) $\sim$ is reflective ($\forall x \in X, x \sim x$), transitive and symmetric ($\forall x, y \in X, x \sim y \Leftarrow y \sim x$).

Definition 4 Utility Functions
A **Utility Function** $u(x)$ assigns a numerical value to each element in X, ranking the elmenst fo X in acordance with the individual's proferences. $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succeq$ if,

$$\forall x, y \in X, x \succeq y \Leftrightarrow u(x) \geq u(y)$$

The utility function that represents apreference relation is not unique. it is only the ranking of alternatives that matters. The preference relation associated with a utility function is an **Ordinal** property (invariant for any strictly increasing transformation). The numerical values associated with the alternatives in X, and hence the magnitude of any differences in the utility measure between alternatives, are **Cardinal** properties (does not preserve under all strictly increasing transformation).

Theorem 5 A preference relation $\succsim$ can be represented by a utility function only if it is rational.

1.2. Choice Rules

In the second approach to the theory of decision making, choice behavior is represented by means of a choice structure.

Definition 6 Choice Structure
A **Choice Structure** $\big(\mathscr{B}, C(\cdot)\big)$ consists of two ingredients:

(1) $\mathscr{B}$ is a family (a set) of nonempty subsets of $X(\mathscr{B} \subseteq 2^X)$. The elements $B \in \mathscr{B}$ are budget sets.

(2) $C(\cdot)$ is a choice rule that assigns a nonempty set of chosen elements $C(B) \subseteq B$ for every budget set $B \in \mathscr{B}$. When $C(B)$ contains a single element, that element is the individual's choice from among the alternatives in B. When $C(B)$ contains multiple elements, the elements of $C(B)$ are the acceptable alternatives in B that the decision maker might choose.

Definition 7 The Weak Axiom of Revealed Preference

The choice structure $(\mathscr{B}, C(\cdot))$ satisfies **The Weak Axiom of Revealed Preference** if the following property holds:

$$\text{If } \exists B \in \mathscr{B} \text{ with } x, y \in B \ s.t. \ x \in C(B),$$
$$\text{then } \forall B' \in \mathscr{B} \text{ with } x, y \in B' \text{ and } y \in C(B'), \text{ we have } x \in C(B').$$

The weak axiom of revealed preference says that if x is ever chosen when y is available, then there can be no budget set containing both alternatives for which y is chosen and x is not. If x is revealed at least as good as y, then y cannot be revealed preferred to x.

Definition (Houthaker's Axiom of Revealed Preferences)

Revealed preference choice rule $C_R : 2^X \to 2^X$ satisfies HARP if whenever $\forall a, b \in X$ and $\forall A, B \subseteq X$,

- $\{a, b\} \subseteq A$ and $a \in C_R(A)$; and
- $\{a, b\} \subseteq B$ and $b \in C_R(B)$,

then we must have $a \in C_R(B)$ (and $b \in C_R(A)$).

Figure 1: HARP.

Definition (Weak Axiom of Revealed Preferences)

Revealed preferences $\widehat{C}_R : \mathcal{B} \to 2^X$ defined only for choice sets $\mathcal{B} \subseteq 2^X$ satisfies WARP if whenever $\forall a, b \in X$ and $\forall A, B \in \mathcal{B}$,

- $\{a, b\} \subseteq A$ and $a \in \widehat{C}_R(A)$; and
- $\{a, b\} \subseteq B$ and $b \in \widehat{C}_R(B)$,

then we must have $a \in \widehat{C}_R(B)$ (and $b \in \widehat{C}_R(A)$).

Figure 2: WARP.

HARP is WARP with all possible choice sets ($\mathcal{B} = 2^X$). HARP is necessary and sufficient for rationalizability. WARP is necessary but not sufficient for rationalizability.

Consider two different budget sets: $\{a, b\}$ and $\{a, b, c\}$. Suppose we know revealed preference choice rules: $C_R(\{a, b\})$ and $C_R(\{a, b, c\})$. We ask: "Is it possible to rationalize this $C_R(\cdot)$?"

- $C_R(\{a, b\}) = \{a\}$ and $C_R(\{a, b, c\}) = \{c\}$
  - Yes: $c \succ a \succ b$
- $C_R(\{a, b\}) = \{a\}$ and $C_R(\{a, b, c\}) = \{a\}$
  - Yes: $a \succ b, a \succ c, b?c$
- $C_R(\{a, b\}) = \{a, b\}$ and $C_R(\{a, b, c\}) = \{c\}$
  - Yes: $c \succ a \sim b$
- $C_R(\{a, b\}) = \{\varnothing\}$ and $C_R(\{a, b, c\}) = \{c\}$
  - No: see Theorem 1
- $C_R(\{a, b\}) = \{b\}$ and $C_R(\{a, b, c\}) = \{a\}$
  - No: Contradiction
- $C_R(\{a, b\}) = \{a\}$ and $C_R(\{a, b, c\}) = \{b, c\}$
  - No: Contradiction

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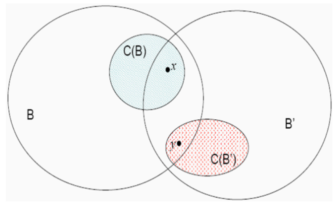
Violation of HARP

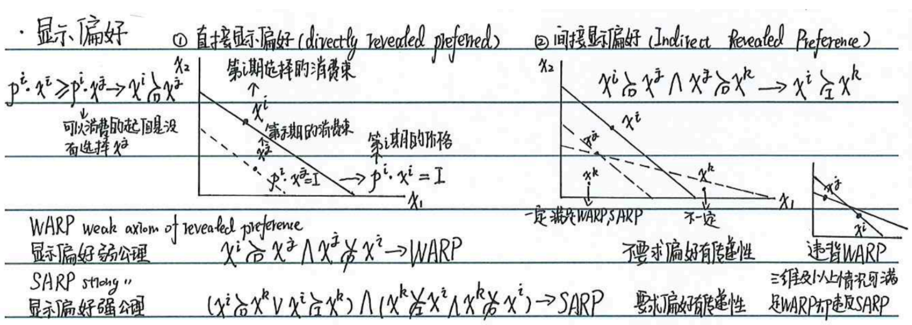


WARP is not sufficient for rationalizability

Example

Consider $\widehat{C}_R : \mathcal{B} \to 2^{\{a,b,c\}}$ defined for choice sets $\mathcal{B} \equiv \{\{a,b\},\{b,c\},\{c,a\}\} \subseteq 2^{\{a,b,c\}}$ with:
$\widehat{C}_R(\{a,b\}) = \{a\}$ (i.e., $a \succ_c b$),
$\widehat{C}_R(\{b,c\}) = \{b\}$ (i.e., $b \succ_c c$),
$\widehat{C}_R(\{c,a\}) = \{c\}$ (i.e., $c \succ_c a$).

$\widehat{C}_R(\cdot)$ satisfies WARP, but is not rationalizable (violates transitivity).



Theorem 8 Suppose $\succeq$ is complete and transitive and B is finite and non-empty, then $C(B, \succeq) \neq \emptyset$.

If B is infinite, then $C(B, \succeq)$ might be empty.

Example

Let $X = [0, \infty)$ and $B \subset X$ is $B = \{1, 2, 3, \ldots\}$. If you prefer more to less ($x \succ y$ if $x > y$), then $C(B, \succeq) = \varnothing$.

Example

Let $B = [0, 1)$. Again, if you prefer more to less ($x \succ y$ if $x > y$), then $C(B, \succeq) = \varnothing$.

Figure 3: E.g.

1.3. The Relationship between Preference Relations and Choice Rules

Proposition 9 Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$ satisfies the weak axiom.

Definition 10 Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ *rationalizes* $C(\cdot)$ relative to $\mathscr{B}$ if

$$C(B) = C^*(B, \succsim)$$

for all $B \in \mathscr{B}$, that is, if $\succsim$ generates the choice structure $(\mathscr{B}, C(\cdot))$.

Proposition 11 Arrow Theorem
If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

1. the weak axiom is satisfied,

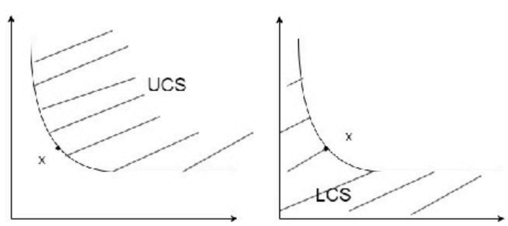
2. $\mathscr{B}$ includes all subsets of X of up to three elements,

then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$; that is, $C(B) = C^*(B, \succsim)$, for all $B \in \mathscr{B}$. Furthermore, this rational preference relation is the *only* preference relation that does.

1.4. Properties of Preferences

Definition (continuous preference relation)

A preference relation $\succeq$ is continuous iff for all $x \in X$, the upper and lower contour sets of x
$\text{UCS}(x) \equiv \{\xi \in X : \xi \succeq x\}$
$\text{LCS}(x) \equiv \{\xi \in X : x \succeq \xi\}$
are both closed sets.

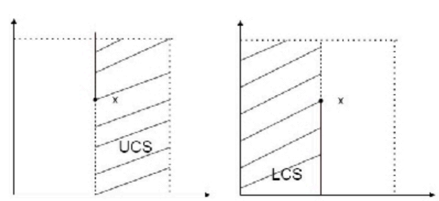


Example (lexicographic preferences)

Preferences over $[0,1]^2 \subset \mathbb{R}^2$ with $(x_1, y_1) \succeq (x_2, y_2)$ iff $x_1 > x_2$, or $x_1 = x_2$ and $y_1 \geq y_2$.

Example (lexicographic preferences are not continuous)

For lexicographic preferences, UCS and LCS are not closed.



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Definition (continuous preference relation)

A preference relation $\succeq$ on X is continuous iff for any sequence of pairs of elements $\{(x_n, y_n)\}_{n=1}^{\infty}$ with $x_n \succeq y_n$ for all n,

$$\lim_{n\to\infty} x_n \succeq \lim_{n\to\infty} y_n.$$

Example (lexicographic preferences are not continuous)

Consider a sequence $x_n = \left(\frac{1}{n}, 0\right)$ and $y_n = (0, 1)$. For every n, we have $x_n \succ y_n$. But $\lim_{n\to\infty} y_n = (0, 1) \succ (0, 0) = \lim_{n\to\infty} x_n$.

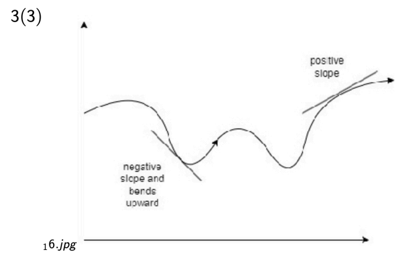
Definition 12 Monotonicity

$\succeq$ satisfies **Monotonicity** at bundle y iff $\forall x \in X$:

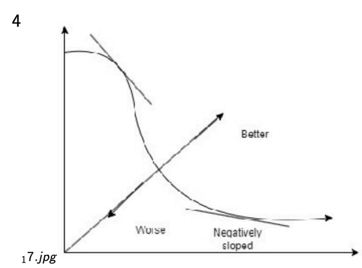
(1) Monotone: $\forall k, x_k \geq y_k \Rightarrow x \succeq y$.

(2) Strictly Monotone: $\forall k, x_k > y_k \Rightarrow x \succ y$.

(3) Strongly Monotone: $\forall k, x_k \geq y_k \wedge x \neq y \Rightarrow x \succ y$.



- Not strictly monotone

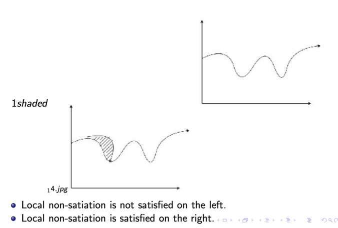


- These preferences satisfy strict monotonicity.
- Indifference curves are negatively sloped.

Definition 13 Local Non-Satiation

$\succeq$ is **Locally Non-satiated** iff $\forall x, \forall \epsilon > 0, \exists y\ s.t.\ \|x - y\| \leq \epsilon\ \big(\exists y \in B_\epsilon(x)\big)$ and $y \succ x$.

If x is strictly monotone, then it is locally non-satiated.



Definition 14 Convex Preferences

(1) $\succeq$ is **Convex** iff

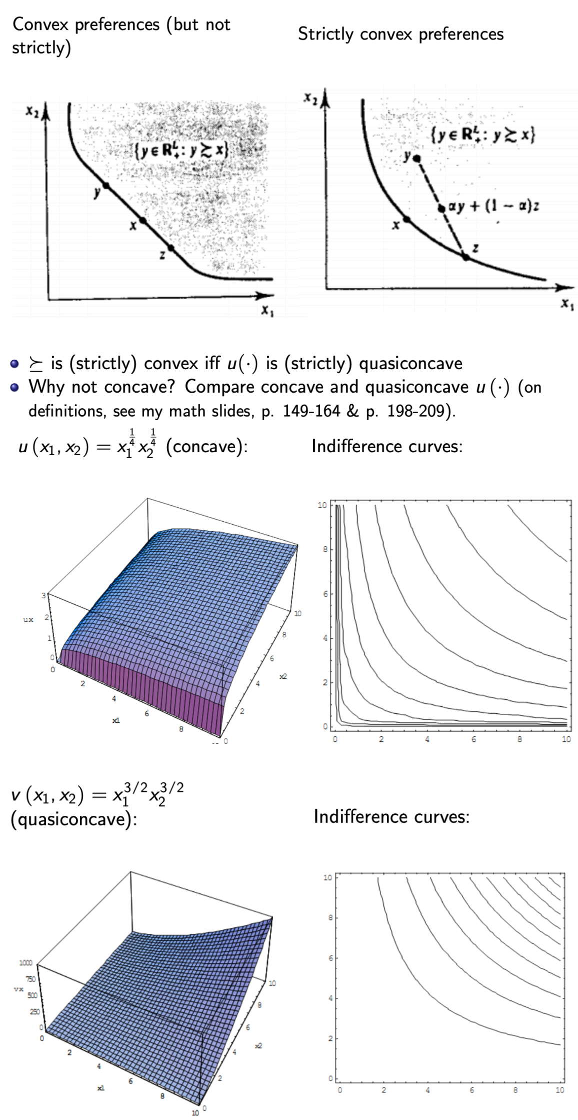
$$y \succeq x \wedge z \succeq x \Rightarrow \forall \lambda \in (0,1), \lambda y + (1-\lambda) z \succeq x$$

(2) $\succeq$ is **Convex** iff the upper contour set of any x is a convex set.

(3) $\succeq$ is **Strictly Convex** iff

$$y \succeq x \wedge z \succeq x \Rightarrow \forall \lambda \in (0,1), \lambda y + (1-\lambda) z \succ x$$

(4) Convex preferences capture the idea that people like diversity. Convexity prohibits the agent from preferring extremes in consumption.



Concavity is not preserved under monotonic transformations.

85 **Definition 15 Equivalent Utility Functions**

- To check whether two utility functions $u(x,y)$ and $v(x,y)$ represent the same preferences, one might look at a marginal rate of substitution:

$$MRS^u_{x,y} = \frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y} = MRS^v_{x,y}$$

- If $MRS^u_{x,y} \neq MRS^v_{x,y}$, then $u(x,y)$ and $v(x,y)$ represent different preferences.

Property of $\succeq$	Property of $u(\cdot)$
– Monotone $(x \geq y \Longrightarrow x \succeq y)$	– Nondecreasing $(x \geq y \Longrightarrow u(x) \geq u(y))$
– Strictly monotone $(x > y \Longrightarrow x \succ y)$	– Increasing $(x > y \Longrightarrow u(x) > u(y))$
– Locally non-satiated	– Has no local maxima in X
– Convex	– Quasiconcave
– Strictly convex	– Strictly quasiconcave

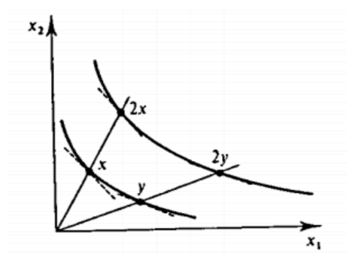
86 **Definition 16 Homotheticity**

87 $\succeq$ is homothetic iff $\forall x, y$ and $\forall \lambda > 0$:

$$x \succeq y \Leftrightarrow \lambda x \succeq \lambda y$$

88

- Continuous, strictly monotone $\succeq$ is homothetic iff it can be represented by a utility function that is homogeneous of degree one (h. of d. 1), i.e., $u(\lambda x) = \lambda u(x)$ for all $\lambda > 0$.
- Homothetic $\succeq$ can also be represented by utility function that isn't h.of d. 1.



89 **Definition 17 Separability**

90 Preferences over x do not depend on y, i.e. $\forall x, x' \in X$ and $\forall y_1, y_2 \in Y$:

$$(x', y_1) \succeq (x, y_1) \Leftrightarrow (x', y_2) \succeq (x, y_2)$$

Example

$u(x_1, x_2, x_3, x_4) = U(v(x_1, x_2), x_3, x_4)$. The preferences are separable on the commodity group $\{1, 2\}$.

$$MRS^u_{x_1,x_2} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{[\partial U/\partial v] \cdot [\partial v/\partial x_1]}{[\partial U/\partial v] \cdot [\partial v/\partial x_2]} = \frac{\partial v/\partial x_1}{\partial v/\partial x_2} = f(x_1, x_2)$$

and it does not depend on the commodity group $\{3, 4\}$. Note that $MRS^u_{x_3,x_4}$ is

$$MRS^u_{x_3,x_4} = \frac{\partial u/\partial x_3}{\partial u/\partial x_4} = \frac{\partial U(v(x_1, x_2), x_3, x_4)/\partial x_3}{\partial U(v(x_1, x_2), x_3, x_4)/\partial x_4} = f(x_1, x_2, x_3, x_4)$$

An example is

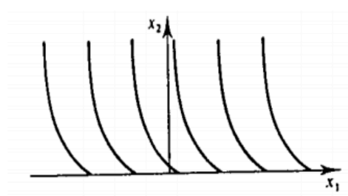
$$u(x_1, x_2, x_3, x_4) = \sqrt{x_1 x_2} \cdot (x_3 \cdot x_4) + x_3^2 + x_4^2.$$

Definition 18 Quasi-linearity

The following $u(x_1, x_2)$ is linear in x_1 and non-linear in x_2:

$$u(x_1, x_2) = x_1 + v(x_2)$$

Each indifference curve is a horizontally shifted copy of the others.



2. Classical Demand Theory

2.1. The Utility Maximization Problem

- The set of optimal choices:

$$\begin{aligned} x(p,w) &\equiv \arg \max_{x \in \mathbb{R}^n_+ : \, p \cdot x \leq w} u(x) \\ &\equiv \arg \max_{x \in B(p,w)} u(x) \\ &= \{x \in \mathbb{R}^n_+ : p \cdot x \leq w \text{ and } u(x) = v(p,w)\} \end{aligned}$$

- *Solution:* $x(p,w)$ – a *Marshalian (Walrasian)* demand;

- $v(p,w)$ is called an *indirect utility function* (as opposed to direct utility u).
- $v(p,w)$ is a value function.

To recover the choice correspondence from the value function we typically apply an Envelope Theorem.

- Value function (indirect utility): $v(p,w) \equiv \sup_x u(x)$ s.t. $px \leq w$.
- Lagrangian: $\mathcal{L} \equiv u(x) + \lambda(w - \sum_i p_i x_i) + \mu \cdot x$.

By the Envelope Theorem, we have

$$1) \ \frac{\partial v}{\partial w} = \frac{\partial \mathcal{L}}{\partial w} = \lambda,$$

$$2) \ \frac{\partial v}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = -\lambda x_i.$$

We can combine 1) and 2), dividing the second by the first.

Roy's identity

$$x_i(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}}.$$

- Roy's identity allows us to recover the Marshallian demand x from the indirect utility v.

Proposition 19 If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

Proposition 20 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on a consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p,w)$ possesses the following properties:

1. *Homogeneity of degree zero in* (p,w)*:* $x(\alpha p, \alpha w) = x(p,w)$ for any p, w and scalar α.

2. *Walras' law:* $p \cdot x = w$ for all $x \in x(p,w)$.

3. *Convexity/uniqueness:* If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if $\succsim$ is *strictly convex*, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

Proposition 21 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The indirect utility function $v(p, w)$ is

1. Homogeneous of degree zero.
2. Strictly increasing in w and nonincreasing in p_ℓ for and ℓ.
3. Quasiconvex; that is, the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.
4. Continuous in p and w.

2.2. The Expenditure Minimization Problem

Expenditure Minimization Problem

$$\min_{x \geq 0} p \cdot x \text{ such that } u(x) \geq \bar{u}.$$

i.e., find the cheapest bundle at price p that yields utility at least $\bar{u}$.

- The solution to the EMP is

$$h(p, \bar{u}) \equiv \arg \min_{x \in \mathbb{R}^n_+ : \, u(x) \geq \bar{u}} p \cdot x,$$

  where $h(p, \bar{u})$ denotes a Hicksian (compensated) demand correspondence, $h : \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^n$. This is a decision function.
- The minimized value of expenditure is

$$e(p, \bar{u}) \equiv \inf_{x \in \mathbb{R}^n_+ : \, u(x) \geq \bar{u}} p \cdot x,$$

  where $e(p, \bar{u})$ is an expenditure function. This is a value function.

Proposition 22 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We have

1. If x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EMP is exactly w.
2. If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this UMP is exactly u.

Proposition 23 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the expenditure function $e(p, u)$ is

1. Homogeneous of degree one in p.
2. Strictly increasing in u and nondecreasing in p_ℓ for any ℓ.
3. Concave in p.
4. Continuous in p and u.

Proposition 24 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possesses the following properties:

1. *Homogeneity of degree zero in* p: $h(\alpha p, u) = h(p, u)$ for any p, u and $\alpha > 0$.
2. *No excess utility:* For any $x \in h(p, u), u(x) = u$.
3. *Convexity/uniqueness:* If $\succsim$ is convex, then $h(p, u)$ is a convex set; and if $\succsim$ is *strictly* convex, so that $u(\cdot)$ is strictly quasiconcave, then there is a unique element in $h(p, u)$.

Proposition 25 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p, u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function $h(p, u)$ satisfies the compensated law of demand: For all p' and p'',

$$(p'' - p') \cdot [h(p'', u) - h(p', u)] \leq 0.$$

2.3. Duality

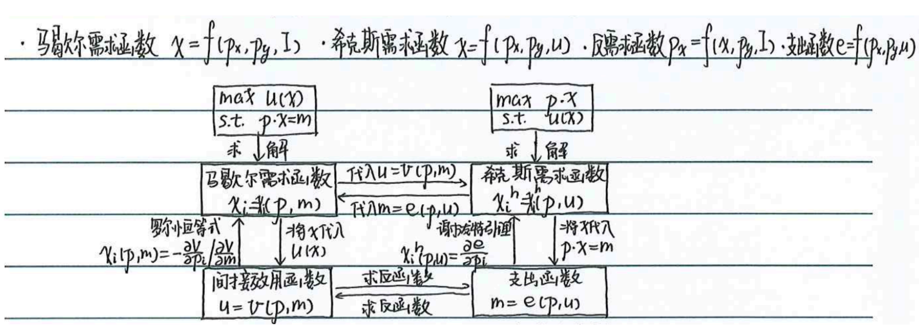
Definition 26 For any nonempty closed set $K \subset \mathbb{R}^L$, the *support function* of K is defined for any $p \in \mathbb{R}^L$ to be

$$\mu_K(p) = \inf\{p \cdot x : x \in K\}.$$

Proposition 27 (The Duality Theorem) Let K be a nonempty closed set, and let $\mu_K(\cdot)$ be its support function. Then there is a unique $\bar{x} \in K$ such that $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$ if and only if $\mu_K(\cdot)$ is differentiable at $\bar{p}$. Moreover, in this case,

$$\nabla \mu_K(\bar{p}) = \bar{x}.$$

2.4. Relationships between Demand, Indirect Utility, and Expenditure Functions



Proposition 28 Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. For all p and u, the Hicksian demand $h(p,u)$ is the derivative vector of the expenditure function with respect to prices:

$$h(p,u) = \nabla_p e(p,u).$$

That is, $h_\ell(p,u) = \partial e(p,u)/\partial p_\ell$ for all $\ell = 1,\ldots,L$.

Proposition 29 (Roy's Identity) Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\bar{p},\bar{w}) \gg 0$. Then

$$x(\bar{p},\bar{w}) = -\frac{1}{\nabla_w v(\bar{p},\bar{w})}\nabla_p v(\bar{p},\bar{w}).$$

That is, for every $\ell = 1,\ldots,L$:

$$x_\ell(\bar{p},\bar{w}) = -\frac{\partial v(\bar{p},\bar{w})/\partial p_\ell}{\partial v(\bar{p},\bar{w})/\partial w}.$$

Proposition 30 (The Slutsky Equation) Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for all (p,w), and $u = v(p,w)$, we have

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{x_\ell(p,w)}{p_k} + \frac{x_\ell(p,w)}{\partial w}x_k(p,w) \quad \text{for all } \ell, k$$

Slutsky equation

$$\underbrace{\frac{\partial x_i(p,w)}{\partial p_j}}_{\text{total effect}} = \underbrace{\frac{\partial h_i(p,u(x(p,w)))}{\partial p_j}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_i(p,w)}{\partial w}x_j(p,w)}_{\text{wealth effect}}$$

for all i and j.

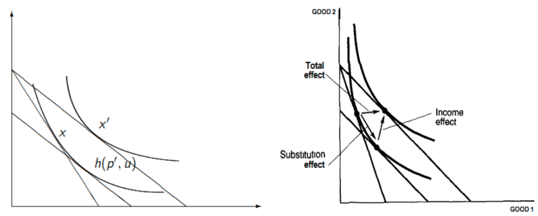
- To see the own-price effect on the Marshallian demand, set $i = j$ in the Slutsky equation:

$$\frac{\partial x_i(p,w)}{\partial p_i} = \frac{\partial h_i(p,v(p,w))}{\partial p_i} - \frac{\partial x_i(p,w)}{\partial w}x_i(p,w).$$

- $p_i \downarrow$ from p_i to p_i':

1. Substitution effect (SE): the consumer is encouraged to consume more of good i ($h_i \uparrow$ from $x_i(p,w) = h_i(p,u)$ to $h_i(p',u)$).
   - Always $\frac{\partial h_i}{\partial p_i} \leq 0$.
2. Wealth effect (WE): the consumer feels richer, which affects x_i in some (indeterminate) way ($x_i \uparrow$ or $\downarrow$ from $h_i(p',u)$ to $x_i(p',u')$).
   - Sign of $\frac{\partial x_i}{\partial w}$ depends on u (preferences).

– Example: $p_1 \downarrow$
– Substitution effect: movement from x to $h(p', \bar{u})$;
– Wealth effect: movement from $h(p', \bar{u})$ to x';
– Total effect: movement from x to x'.



· 斯拉茨基公式 价格变动前 (p_1, p_2) (x_1, x_2) $\max u(x_1, x_2)$ s.t. $p_1 x_1 + p_2 x_2 = m$ 需求1. $x_1(p_1, p_2, m)$

价格变动后 (p_1', p_2) (x_1', x_2') $\max u(x_1, x_2)$ s.t. $p_1' x_1' + p_2 x_2' = m$ 需求3. $x_1(p_1', p_2, m)$

假设中间步骤, $[p_1 + (p_1' - p_1)]x_1 + p_2 x_2 = m + (p_1' - p_1)x_1$, $\Delta m = m' - m = x_1(p_1' - p_1)$ $\max u(x_1, x_2)$ s.t. $p_1' x_1 + p_2 x_2 = m'$ 需求2. $x_1(p_1', p_2, m')$

SE: $\Delta x_1^s = x_1(p_1', p_2, m') - x_1(p_1, p_2, m)$ IE: $\Delta x_1^n = x_1(p_1', p_2, m') - x_1(p_1', p_2, m)$

$TE = SE - IE = [x_1(p_1', p_2, m') - x_1(p_1, p_2, m)] - [x_1(p_1', p_2, m') - x_1(p_1', p_2, m)] = x_1(p_1', p_2, m) - x_1(p_1, p_2, m)$

$$\frac{\partial x_1(p_1, p_2, m)}{\partial p_1} = \underbrace{\frac{\partial x_1^h(p_1, p_2, \bar{u})}{\partial p_1}}_{SE} - \underbrace{x_1(p_1, p_2, m)\frac{\partial x_1(p_1, p_2, m)}{\partial m}}_{IE}$$

162 2.5. Marshallian Response to Changes in Wealth

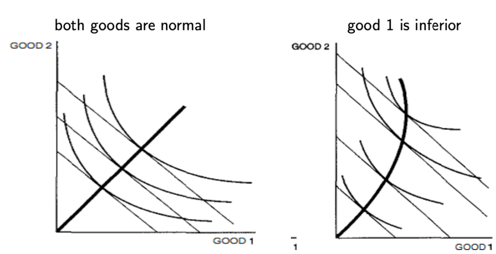
Definition (Normal good)

Good i is a normal good if $x_i(p, w)$ is increasing in w (i.e., $\frac{\partial x_i(p,w)}{\partial w} > 0$).

Definition (Inferior good)

Good i is an inferior good if $x_i(p, w)$ is decreasing in w (i.e., $\frac{\partial x_i(p,w)}{\partial w} < 0$).

- *Engel curve* = wealth expansion path (how x moves with w).



At least one of the goods should be normal (the agent should spend his wealth on something).

163 2.6. Marshallian Response to Changes in Own Price

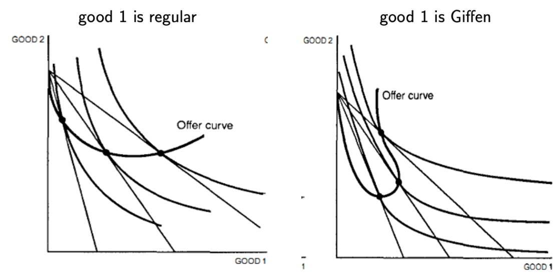
Definition (Regular good)

Good i is a regular good if $x_i(p, w)$ is decreasing in p_i (i.e., $\frac{\partial x_i(p,w)}{\partial p_i} < 0$).

Definition (Giffen good)

Good i is a Giffen good if $x_i(p, w)$ is increasing in p_i (i.e., $\frac{\partial x_i(p,w)}{\partial p_i} > 0$).

Offer curve = price expansion path (how x moves with p).



The law of demand holds for x_i if the good is regular.

- If a good is normal, $\frac{\partial x_i}{\partial w}x_i > 0 \Rightarrow \underbrace{\frac{\partial h_i}{\partial p_i}}_{<0} \overbrace{-\underbrace{\frac{\partial x_i}{\partial w}x_i}_{>0}}^{<0} = \frac{\partial x_i}{\partial p_i} < 0 \Rightarrow$ The good is regular.
- If a good is Giffen, $\frac{\partial x_i}{\partial p_i} > 0, \Longrightarrow \underbrace{\frac{\partial x_i}{\partial p_i}}_{>0} = \underbrace{\frac{\partial h_i}{\partial p_i}}_{<0} - \frac{\partial x_i}{\partial w}x_i \Rightarrow \frac{\partial x_i}{\partial w}x_i < 0 \Rightarrow$ The good is inferior.

164 2.7. Marshallian Response to Changes in Other Goods' Price

Definition (Gross substitute)

Good i is a gross substitute for good j if $x_i(p, w)$ is increasing in p_j (i.e., $\frac{\partial x_i(p,w)}{\partial p_j} > 0$).

Definition (Gross complement)

Good i is a gross complement for good j if $x_i(p, w)$ is decreasing in p_j (i.e., $\frac{\partial x_i(p,w)}{\partial p_j} < 0$).

Gross substitutability/complementarity is not *necessarily symmetric.*

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165 2.8. Hicksian Response to Changes in Other Goods' Price

Definition (Substitute)

Good i is a substitute for good j if $h_i(p, \bar{u})$ is increasing in p_j (i.e., $\frac{\partial h_i(p,w))}{\partial p_j} > 0$).

Definition (Complement)

Good i is a complement for good j if $h_i(p, \bar{u})$ is decreasing in p_j (i.e., $\frac{\partial h_i(p,w)}{\partial p_j} < 0$).

Substitutability/complementarity is symmetric.

- If goods i, j are *substitutes*, $\frac{\partial h_i}{\partial p_j} > 0$, and good i is *inferior*, $\frac{\partial x_i}{\partial w} < 0$,
  $$\Rightarrow \frac{\partial x_i}{\partial p_j} = \underbrace{\frac{\partial h_i}{\partial p_j}}_{>0} - \underbrace{\frac{\partial x_i}{\partial w}}_{<0} x_j \Rightarrow \frac{\partial x_i}{\partial p_j} > 0$$
  $\Rightarrow$ The good i is a *gross substitute* for good j.
- If goods i, j are *complements*, $\frac{\partial h_i}{\partial p_j} < 0$, and good i is *normal*, $\frac{\partial x_i}{\partial w} > 0$,
  $$\Rightarrow \frac{\partial x_i}{\partial p_j} = \underbrace{\frac{\partial h_i}{\partial p_j}}_{<0} - \underbrace{\frac{\partial x_i}{\partial w}}_{>0} x_j \Rightarrow \frac{\partial x_i}{\partial p_j} < 0$$
  $\Rightarrow$ The good i is a *gross complement* for good j.

3. Uncertainty

3.1. Expected Utility Theory

Definition 31 Von Neumann-Morgenstern Expected Utility Model

(1) $\mathcal{X}$ = set of all possible prizes (outcomes or consequences), $\mathcal{X} = \{x_1, ..., x_n\}$. $\mathcal{X}$ can take many forms (e.g.,consumption bundles, monetary payoffs).

(2) $|\mathcal{X}| = n < \infty$. There must be a best outcome and a worst outcome.

(3) A **Simple Lottery** is a probability distribution $p = (p_1, \ldots, p_n) \in \mathbb{R}^n_+$ over prizes, where p_i is the probability that outcome x_i occurs. A simple lottery can be represented geometrically in the $(n-1)$ dimensional simplex:

$$\Delta(\mathcal{X}) \equiv \left\{ p \in \mathbb{R}^n_+ \colon \sum_i^n p_i = 1 \right\}$$

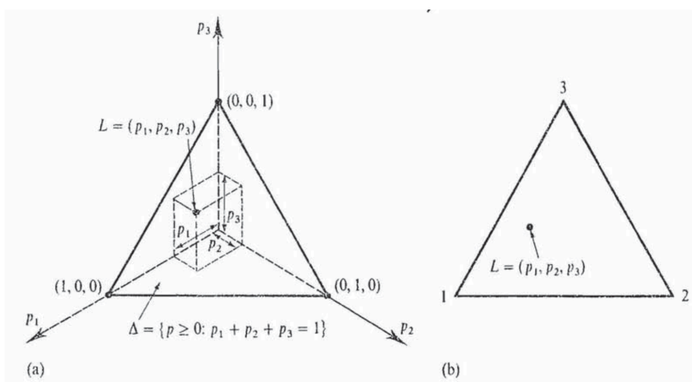


Figure 4: Simple lottery.

A *simple lottery* L is a list $L = (p_1, \ldots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

(4) A **Compound Lottery** allows the outcomes of a lottery themselves to be simple lotteries.

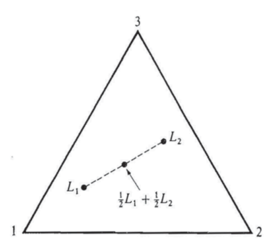


Figure 5: Compound lottery.

Given K simple lotteries $L_k = (p_1^k, \ldots, p_N^k)$, $k = 1, \ldots, K$, and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the *compound lottery* $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, \ldots, K$.

Definition 32 Preferences Over Lotteries

(1) A rational decision-maker has preferences over outcomes in $\mathcal{X}$.

(2) We consider preferences over lotteries $\Delta(\mathcal{X})$. From now on, $\succeq$ refers to preferences over lotteries, not outcomes.

- A.1. $\succeq$ is rational (complete and transitive) on the set of all lotteries over the set of outcomes $\mathcal{X}$.
- A.2. $\succeq$ is continuous.
- A.3. $\succeq$ satisfies independence axiom.

Definition 33 Continuity of Preferences
A preference relation $\succeq$ over $\Delta(\mathcal{X})$ is continuous iff for any p_H, p_M, and $p_L \in \mathcal{X}$ such that $p_H \succeq p_M \succeq p_L$, there exists some $\alpha \in [0, 1]$ such that:

$$\alpha p_H + (1 - \alpha) p_L \sim p_M$$

i.e., if you move slightly away (say in the direction of the worst lottery p_L) from one lottery which you prefer p_H over a second one p_M , at some point, you will be indifferent to the second one.

Definition 34 Independence Axiom
A preference relation $\succeq$ over $\Delta(\mathcal{X})$ satisfies independence iff for any p, p', and $p_m \in \Delta(\mathcal{X})$ and any $\alpha \in [0, 1]$, we have

$$p \succeq p' \quad \Leftrightarrow \quad \alpha p + (1 - \alpha) p_m \succeq \alpha p' + (1 - \alpha) p_m$$

i.e., if p is at least as good as p', then the possibility of p is at least as good as the possibility of p', as long as the other possibility is the same (a $(1 - \alpha)$ chance of p_m) in both cases. Similar relationships hold for $\succ$ and $\sim$.

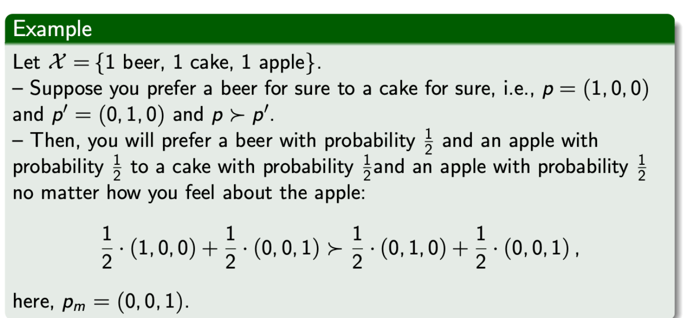


Figure 6: E.g.

Definition 35 Von Neumann-Morgenstern Utility Functions
A utility function $U : \Delta(\mathcal{X}) \to \mathbb{R}$ is a VNM utility function (equivalently, has an expected utility form) iff there exist numbers $u_1, \ldots, u_n \in \mathbb{R}$ such that for every $p \in \Delta(\mathcal{X})$,

$$U(p) = \sum_{i=1}^{n} p_i u_i = p \cdot \vec{u}$$

where $\vec{u} \equiv (u_1, \ldots, u_n) \in \mathbb{R}^n$.

The utility function $U : \mathscr{L} \to \mathbb{R}$ has an *expected utility form* if there is an assignment of numbers $(u_1, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N) \in \mathscr{L}$ we have

$$U(L) = u_1 p_1 + \cdots + u_N p_N.$$

A utility function $U : \mathscr{L} \to \mathbb{R}$ with the expected utility form is called a *von Neumann-Morgenstern (v.N-M) expected utility function.*

Theorem 36 Linearity of VNM Utility Functions
A utility function $U : \Delta(\mathcal{X}) \to \mathbb{R}$ is a VNM utility function iff it is linear, i.e.,it satisfies:

$$U(\alpha p + (1-\alpha)p') = \alpha U(p) + (1-\alpha)U(p')$$

for all p, $p' \in \Delta(\mathcal{X})$, and $\alpha \in [0, 1]$.

A utility function $U : \mathscr{L} \to \mathbb{R}$ has an expected utility form if and only if it is *linear*, that is, if and only if it satisfies the property that

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any K lotteries $L_k \in \mathscr{L}$, $k = 1, \ldots, K$, and probabilities $(\alpha_1, \ldots, \alpha_K) \geq 0, \sum_k \alpha_k = 1$.

Theorem 37 Expected Utility Theorem
Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \ldots, N$ in such a manner that for any two lotteries $L = (p_1, \ldots, p_N)$ and $L' = (p'_1, \ldots, p'_N)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_{n-1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n.$$

Theorem 38 Robust to Affine Transformations
Suppose $U : \Delta(\mathcal{X}) \to \mathbb{R}$ is an expected utility representation of $\succeq$. Then $V : \Delta(\mathcal{X}) \to \mathbb{R}$ is also an expected utility representation of $\succeq$ iff there exist some scalars $a \in \mathbb{R}$ and $b \in \mathbb{R}_{++}$ such that

$$V(p) = a + bU(p)$$

for all $p \in \Delta(\mathcal{X})$.

- A linear representation is not unique. If $U(p)$ is an expected utility representation, we can rescale it, $V(p) = a + bU(p)$, a any and $b > 0$, and obtain another expected utility representation, which is linear.

- We say that an expected utility representation is robust to increasing linear (affine) transformations.

- If $U(p)$ is EUR, then $V(p)$ is EUR under appropriate a and b.

- a and b are not arbitrary but related to $V(\overline{p})$, $V(\underline{p})$ and $U(\overline{p})$, $U(\underline{p})$
- When we go from U to V, we perform a linear change of variables.
- We map each point in the interval $\left[U(\underline{p}), U(\overline{p})\right]$ into a point in the interval $\left[V(\underline{p}), V(\overline{p})\right]$.
- Using $V(p) = a + bU(p)$, we get $a = \frac{V(\underline{p})U(\overline{p}) - U(\underline{p})V(\overline{p})}{U(\overline{p}) - U(\underline{p})}$ and $b = \frac{V(\overline{p}) - V(\underline{p})}{U(\overline{p}) - U(\underline{p})}$

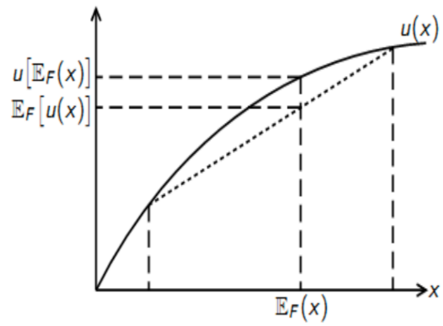
3.2. Money Lotteries and Risk Aversion

- In the **finite** case, a vNM utility function was
$$U(p) = \sum_i p_i u_i.$$
- The **continuous** analogue of a vNM utility function over cdfs is
$$U(F) = \int_{\mathbb{R}} u(x)\, dF(x) \equiv \mathbb{E}_F\left[u(x)\right],$$
where $U\colon \mathbb{F} \to \mathbb{R}$ ("vNM utility function") represents preferences over lotteries,
$u\colon \mathbb{R} \to \mathbb{R}$ ("**Bernoulli utility function**") indexes preference over outcomes.

Definition 39 A decision maker is a *risk averse* (or exhibits *risk aversion*) if for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int x dF(x)$ with certainty is at least as good as the lottery $F(\cdot)$ itself. If the decision maker is always [i.e. for any $F(\cdot)$] indifferent between these two lotteries, we say that he is *risk neutral*. Finally, we say that he is *strictly risk averse* if indifference holds only when the two lotteries are the same [i.e. when $F(\cdot)$ is degenerate].

$$u\left(\mathbb{E}_F[x]\right) \geq \mathbb{E}_F\left[u(x)\right]$$

utility from certain payoff ≥ utility from the lottery



- The inequality

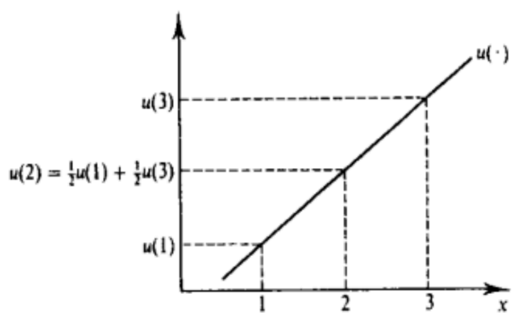
$$u\left(\mathbb{E}_F[x]\right) \geq \mathbb{E}_F\left[u(x)\right]$$
$$u\left(\int_{\mathbb{R}} x\, dF(x)\right) \geq \int_{\mathbb{R}} u(x)\, dF(x)$$

is called *Jensen's inequality*.
- It's a defining property of a concave function.

Theorem (4)

A decision-maker is (strictly) risk-averse iff her Bernoulli utility function is (strictly) concave.

The decision-maker is risk-neutral iff $u(\cdot)$ is linear.



Definition 40 Given a Bernoulli utility function $u(\cdot)$ we defined the following concepts:

1. The *certainty equivalent of* $F(\cdot)$, denoted $c(F, u)$, is the amount of money for which the individual is indifferent between the gamble $F(\cdot)$ and the certain amount $c(F, u)$; that is
$$u\left(c(F, u)\right) = \int u(x) dF(x).$$

2. For any fixed amount of money x and positive number ε, the *probability premium* denoted by $\pi(x, \varepsilon, u)$, is the excess on winning the probability over fair odds that makes the individual indifferent between the certain outcome x and a gamble between the two outcomes $x + \varepsilon$ and $x - \varepsilon$. That is

$$u(x) = \left(\frac{1}{2} + \pi(x, \varepsilon, u)\right) u(x + \varepsilon) + \left(\frac{1}{2} - \pi(x, \varepsilon, u)\right) u(x - \varepsilon).$$

Proposition 41 Suppose a decision maker is an expected utility maximizer with a Bernoulli utility function $u(\cdot)$ on amounts of money. Then the following properties are equivalent:

1. The decision maker is risk averse.
2. $u(\cdot)$ is concave.
3. $c(F, u) \leq \int x dF(x)$ for all $F(\cdot)$.
4. $\pi(x, \varepsilon, u) \geq 0$ for all x, ε.

Definition 42 Given a (twice differentiable) Bernoulli utility function $u(\cdot)$ for money, the *Arrow Pratt coefficient of absolute risk aversion* at x is defined as $r_A(x) = -u''(x)/u'(x)$.

Definition (Arrow-Pratt coefficient)

For a twice differentiable Bernoulli utility function $u(\cdot)$, the Arrow-Pratt coefficient of absolute risk aversion is

$$A_u(x) \equiv -\frac{u''(x)}{u'(x)}.$$

Definition 43 (More-risk-averse-than) Given two Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$, when can we say that $u_2(\cdot)$ is unambiguously *more risk averse than* $u_1(\cdot)$? Several possible approaches to a definition seem plausible:

1. $r_A(x, u_2) \geq r_A(x, u_1)$ for every x.
2. There exists an increasing concave function $\psi(\cdot)$ such that $u_2(x) = \psi(u_1(x))$ at all x; that is, $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$. [In other words, $u_2(\cdot)$ is "more concave" than $u_1(\cdot)$.]
3. $c(F, u_2) \leq c(F, u_1)$ for any $F(\cdot)$.
4. $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.
5. Whenever $u_2(\cdot)$ finds a lottery $F(\cdot)$ at least as good as a riskless outcome $\bar{x}$, then $u_1(\cdot)$ also finds $F(\cdot)$ at least as good as $\bar{x}$. That is, $\int u_2(x)dF(x) \geq u_2(\bar{x})$ implies $\int u_1(x)dF(x) \geq u_1(\bar{x})$ for any $F(\cdot)$ and $\bar{x}$.

Proposition 44 Definitions (i) to (v) of the *more-risk-averse-than* relation are equivalent.

Definition 45 The Bernoulli utility function $u(\cdot)$ for money exhibits *decreasing absolute risk aversion* if $r_A(x, u)$ is a decreasing function of x.

Proposition 46 The following properties are equivalent:

1. The Bernoulli utility function $u(\cdot)$ exhibits decreasing absolute risk aversion.
2. Whenever $x_2 < x_1$, $u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.
3. For any risk $F(z)$, the certainty equivalent of the lottery formed adding risk z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x+z)dF(z)$, is such that $(x - c_x)$ is decreasing in x. That is, the higher x is, the less is the individual willing to pay to get rid of the risk.
4. The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.
5. For any $F(z)$, if $\int u(x_2+z)dF(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int u(x_1+z)dF(z) \geq u(x_1)$.

Definition (decreasing / constant / increasing absolute risk aversion)

The Bernoulli utility function $u(\cdot)$ has decreasing /constant / increasing absolute risk aversion iff $A_u(\cdot)$ is a decreasing / constant / increasing function of x.

Definition 47 Given a Bernoulli utility function $u(\cdot)$, the *coefficient of relative risk aversion at x* is $r_R(x, u) = -xu''(x)/u'(x)$.

Definition (coefficient of relative risk aversion)

Given a twice differentiable Bernoulli utility function $u(\cdot)$,

$$R_u(x) \equiv -x\frac{u''(x)}{u'(x)} = xA_u(x).$$

We can define decreasing / increasing / constant relative risk aversion as above, but using $R_u(\cdot)$ instead of $A_u(\cdot)$.

Proposition 48 The following conditions for a Bernoulli utility function $u(\cdot)$ on amounts of money are equivalent:

1. $r_R(x, u)$ is decreasing in x.
2. Whenever $x_2 < x_1$, $\tilde{u}_2(t) = u(tx_2)$ is a concave transformation of $\tilde{u}_1(t) = u(tx_1)$.
3. Given any risk $F(t)$ on $t > 0$, the certainty equivalent $\bar{c}_x$ defined by $u(\bar{c}_x) = \int u(tx)dF(t)$ is such that $x/\bar{c}_x$ is decreasing in x.

3.3. Comparison of Payoff Distributions in Terms of Return and Risk

Definition 49 The distribution $F(\cdot)$ *first-order stochastically dominates* $G(\cdot)$ if, for every nondecreasing function $u : \mathbb{R} \to \mathbb{R}$, we have

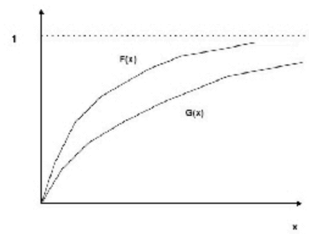
$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Proposition 50 The distribution of monetary payoffs $F(\cdot)$ first-order stochastically dominates the distribution $G(\cdot)$ if and only if $F(x) \leq G(x)$ for every x.

Theorem (6)

Distribution G first-order stochastically dominates distribution F iff $G(x) \leq F(x)$ for all x.

i.e., lottery G is more likely than F to pay at least x for any threshold x.



Definition 51 For any two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ *second-order stochastically dominates* (or *is less risky than*) $G(\cdot)$ if for every nondecreasing concave function $u : \mathbb{R}_+ \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Theorem (6)

Distribution G second-order stochastically dominates distribution F iff

$$\int_{-\infty}^{x} G(t)\, dt \leq \int_{-\infty}^{x} F(t)\, dt \text{ for all } x.$$

i.e., for all x, the **area** under G is smaller or equal than the area under F.

Proposition 52 Consider two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean. Then the following statements are equivalent:

1. $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.
2. $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.
3. Property 51 holds.

- In the definition of SOSD, G and F are assumed to have the same mean. This assumption is made to prove the theorem about SOSD.
- If they G and F do not have the same mean, we can still compare them in terms of SOSD.
- **Result:** If G FOSD $F \implies G$ SOSD F (left without a proof).
- The converse should not be true.

295 3.4. Insurance

- Consider a *strictly* risk-averse agent (i.e., $u'' < 0$).
- His endowment of wealth is $\$w$.
- Suppose that there is just one state of the world: with probability p he can loose $\$L$.
- He can ensure himself against this loss by purchasing insurance.
- Each unit of insurance costs $\$q$.
- If the amount of insurance bought is a, the total cost is $\$qa$.
- In the case of loss, each unit of insurance pays 1\$ (total payment is $\$a$) and nothing otherwise.

The agent maximizes expected utility:

$$\max_a U(a) \equiv \{pu[w - qa - L + a] + (1 - p)u[w - qa]\}.$$

What if insurance is actuarially fair?

- That is, the insurance company makes zero-profit. Suppose the insurance company solves:
$$\max_a [qa - pa].$$
Then, the FOC wrt a is: $q = p$ (i.e., the company sets price of insurance equal to probability), and profit is zero.
- The agent's FOC becomes
$$\begin{aligned}(1-p)pu'[w - qa^* - L + a^*] &= p(1-p)u'[w - qa^*]\\ u'[w - qa^* - L + a^*] &= u'[w - qa^*]\\ w - qa^* - L + a^* &= w - qa^*\\ a^* &= L\end{aligned}$$
- That is, the agent fully insures himself against risk of loss.

What if insurance is not actuarially fair?

- Suppose cost of insurance is above expected loss: $q > p$.
- $q > p \Rightarrow (1-q) < (1-p)$.
- FOC is
$$\begin{aligned}\frac{u'[w - qa^* - L + a^*]}{u'[w - qa^*]} &= \frac{q(1-p)}{p(1-q)}\\ &> 1\\ u'[w - qa^* - L + a^*] &> u'[w - qa^*]\\ &\text{– Since } u' \text{ is decreasing } (u'' < 0)\text{:}\\ w - qa^* - L + a^* &< w - qa^*\\ a^* &< L.\end{aligned}$$

- The agent underinsures against risk of loss. Why? It's too costly to transfer wealth to the loss state, so he transfers less than L.

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4. Producer Theory

4.1. Assumptions

A.1 Firms are price takers

A.2 Technology is exogenously given.

A.3 Firms maximize profits.

4.2. Production Sets

Definition 53 Production Plan

Definition (production plan)

A vector $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ where an output has $y_k > 0$ and an input has $y_k < 0$.

Note y is a net output vector.

Example

Let $(3, 2, 0, 0)$ be a vector of inputs, and $(0, 2, 0, 4)$ be a vector of outputs. Then, $y = (-3, 0, 0, 4)$.

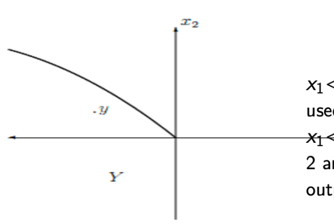
Example (continued)

If the prices of these goods are $p = (1, 2, 1, 2)$, then a firm earns profit of $p \cdot y = (1, 2, 1, 2) \cdot (-3, 0, 0, 4)^\top = 5$.

Definition 54 Production Set

Definition (production set)

Set $Y \subseteq \mathbb{R}^n$ of **feasible** production plans; generally assumed to be non-empty and closed.



$x_1 < 0$ and $x_2 > 0$: good 1 is used to produce good 2.
$x_1 < 0$ and $x_2 < 0$: goods 1 and 2 are used without producing any output.

$Y = \{(0,0), (-3,9), (-1,18)\}$,
$Y = \{x_2 + kx_1 \leq 0,\ x_1 \leq 0,\ k > 0\}$.

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Definition (shutdown)

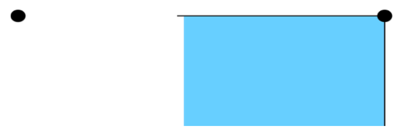
$\mathbf{0} \in Y$.

Can produce nothing (no inputs, no outputs).
(In short-run, sometimes cannot do it quickly but in the long-run, yes)

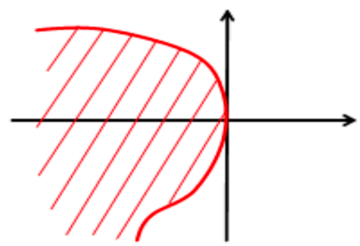
Definition (free disposal)

$y \in Y$ and $y' \leq y$ imply $y' \in Y$.

Can throw away any (continuous) amount of output or input



Violation of free disposal



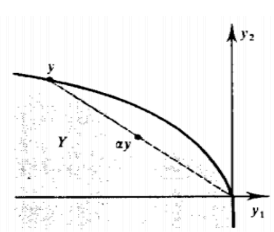
Proposition 55 The production set Y is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.

Definition (nonincreasing returns to scale)

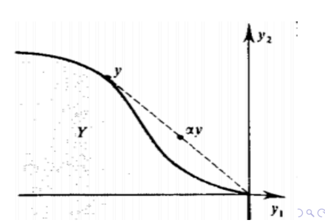
$y \in Y$ implies $\alpha y \in Y$ for all $\alpha \in [0, 1]$.

– Any feasible y can be scaled down (no advantage from scaling production up).
– Implies shutdown (set $\alpha = 0$).

Nonincreasing returns is satisfied:



Nonincreasing returns is not satisfied:



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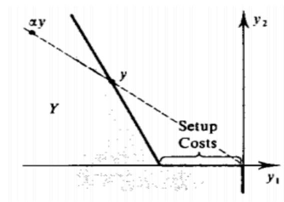
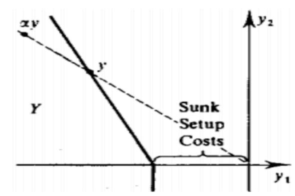
Definition (nondecreasing returns to scale)

$y \in Y$ implies $\alpha y \in Y$ for all $\alpha \geq 1$.

Can scale up (i.e., replicate).



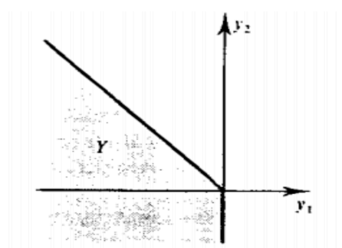
Setup costs ($\mathbf{0} \in Y \Rightarrow$ can recover costs, i.e., not sunk).



Sunk costs ($\mathbf{0} \notin Y$).

Definition (constant returns to scale)

$y \in Y$ implies $\alpha y \in Y$ for all $\alpha \geq 0$; i.e., nonincreasing *and* nondecreasing returns to scale.

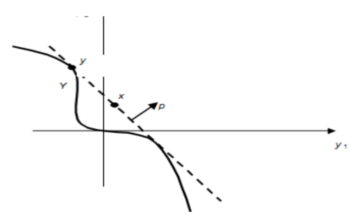


306 Definition 56 Convex Production Set

Definition (convex production set)

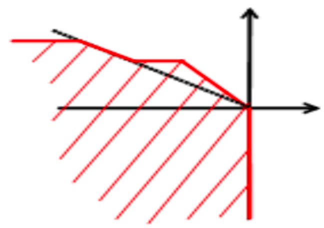
$y, y' \in Y$ imply $ty + (1-t)y' \in Y$ for all $t \in [0, 1]$.

- **Strictly convex** iff for $t \in (0, 1)$, the convex combination is in the *interior* of Y
- Nonconvex production set:



Implications of the above assumptions:

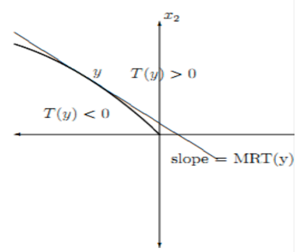
- If $\mathbf{0} \in Y$, then convexity $\Longrightarrow$ nonincreasing returns to scale.
- Nonincreasing returns to scale does not imply convexity.



307 **Definition 57 Transformation Function**

Definition (transformation function)
Any function $T\colon \mathbb{R}^n \to \mathbb{R}$ such that $Y = \{y \in \mathbb{R}^n : T(y) \leq 0\}$.

1. $T(y) > 0 \iff y$ is outside of Y;
2. $T(y) = 0 \iff$ y is on the frontier Y;
3. $T(y) < 0 \iff y$ is in the interior of Y (there is some waste with y).



Definition (transformation frontier)
The set $\{\in \mathbb{R}^n \colon T(y) = 0\}$ (or production possibilities frontier).

If there is technological progress, the frontier is growing.

- production function;
- production set;
- transformation frontier.

Example (Cobb-Douglas function)
Alternative ways to define Cobb-Douglas technology with two inputs $y_2 < 0$ and $y_3 < 0$ (here, $y = (y_1, y_2, y_3)$):
1. $y_1 = (-y_2)^{1/2}(-y_3)^{1/2}$ (production function);
2. $Y = \left\{(y_1, y_2, y_3) \text{ in } \mathbb{R}^3 : y_1 \leq (-y_2)^{1/2}(-y_3)^{1/2}\right\}$ (production set);
3. $T(y_1, y_2, y_3) = y_1 - (-y_2)^{1/2}(-y_3)^{1/2} = 0$ (transformation frontier).

308 **4.3. Profit Maximization**

309 **Definition 58 Returns to Scale**

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- Suppose there is just one firm: $Y = F(K, N) = K^{\alpha} N^{\beta}$ with $\alpha + \beta > 0$, where K=total capital, N=total labor.
- Suppose we split this firm into $x > 1$ smaller firms. Will x smaller firms produce more or less output?
- Let each firm gets xth part of K and N: $k = \frac{K}{x}$ and $n = \frac{N}{x}$.
- Denote by y output of a small firm. Each small firm uses the same technology.
- $Y' \equiv y \cdot x = k^{\alpha} n^{\beta} \cdot x = \left(\frac{K}{x}\right)^{\alpha} \left(\frac{N}{x}\right)^{\beta} \cdot x = x^{1-\alpha-\beta} \cdot K^{\alpha} N^{\beta} = x^{1-\alpha-\beta} \cdot Y$
- Is $Y' \gtrless Y$?
- If $\alpha + \beta = 1$, we have CRTS so that $Y' = Y \Rightarrow$ The number of firms is not important for Y.
- If $\alpha + \beta > 1$, we have IRTS, so that $Y' < Y \Rightarrow$It pays to be a big firm.
- If $\alpha + \beta < 1$, we have DRTS, so that $Y' > Y \Rightarrow$It pays to be a small firm.

310 **Definition 59 Marginal Rate of Transformation (MRT/MRTS)**

When the transformation function is differentiable, we can define the marginal rate of transformation of good l for good k.

Definition (marginal rate of transformation)

$$\mathrm{MRT}_{l,k}(y) \equiv \frac{\frac{\partial T(y)}{\partial y_l}}{\frac{\partial T(y)}{\partial y_k}},$$

defined for points where $T(y) = 0$ and $\frac{\partial T(y)}{\partial y_k} \neq 0$.

Measures the extra amount of good k that can be obtained per unit reduction of good l.

- To obtain MRT, we start from $T(\bar{y}) = 0$.
- Find a full differential from two sides:

$$\frac{\partial T(\bar{y})}{\partial y_k} dy_k + \frac{\partial T(\bar{y})}{\partial y_l} dy_l = 0,$$

- Thus,

$$\mathrm{MRT}_{l,k}(\bar{y}) = -\frac{dy_k}{dy_l} = \frac{\partial T(\bar{y}) / \partial y_l}{\partial T(\bar{y}) / \partial y_k} = -\text{ Slope of } T(\bar{y})$$

311 **Definition 60 Elasticity of Substitution**

- MRTS is one local measure of substitutability between inputs in producing a given level of output.
- To measure substitutability, economists use elasticity of substitution, σ, which is unit free.
- Between two inputs, z_i and z_j, holding all other inputs and the level of output constant, the elasticity is defined as the percentage change in the input proportions, z_j/z_i, associated with a 1 percent change in the MRTS between them.
- Formally,

$$\begin{aligned}\sigma_{ij} &= \frac{d\ln\left(z_j/z_i\right)}{d\ln\left(\frac{\partial f(z)/\partial z_i}{\partial f(z)/\partial z_j}\right)} = \frac{d\ln\left(z_j/z_i\right)}{d\ln\left(MRTS_{i,j}\right)} \\ &= \frac{d\left(z_j/z_i\right)}{z_j/z_i} / \frac{d\left(MRTS_{i,j}\right)}{MRTS_{i,j}} = \frac{d\left(z_j/z_i\right)}{dMRTS_{i,j}} \cdot \frac{MRTS_{i,j}}{z_j/z_i}.\end{aligned}$$

Example (Elasticity)

$f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$ (production function) with $\alpha > 0$ and $\beta > 0$.
$\frac{\partial f(z)}{\partial z_1} = \alpha z_1^{\alpha-1} z_2^{\beta}$
$\frac{\partial f(z)}{\partial z_2} = \beta z_1^{\alpha} z_2^{\beta-1}$.
$\mathrm{MRTS}_{2,1}(\overline{q}, \overline{z}) = \frac{\beta z_1^{\alpha} z_2^{\beta-1}}{\alpha z_1^{\alpha-1} z_2^{\beta}} = \frac{\beta}{\alpha}\frac{z_1}{z_2} > 0.$
$\sigma_{2,1} = \frac{d(z_1/z_2)}{z_1/z_2} / \frac{dMRTS_{2,1}}{MRTS_{2,1}} = 1/\left(\frac{dMRTS_{2,1}}{d(z_1/z_2)}\right) \cdot \frac{MRTS_{2,1}}{z_1/z_2} = \frac{\alpha}{\beta}\frac{\beta}{\alpha} = 1.$

312 Definition 61 Profit Maximization

Given price vector p and production set Y, find optimal y:

$$\max_y \; p \cdot y \quad y \in Y \qquad \text{or} \qquad \max_y \; p \cdot y \quad \text{s.t. } T(y) \leq 0.$$

– Solve this problem for different $p \Rightarrow$ Get an optimal supply correspondence $Y^*(p)$ ($Y^*(p) \equiv \arg\max_{y \in Y} p \cdot y$).
– After $Y^*(p)$ is found, compute profit at optimum: $\pi(p) = pY^*(p)$ (this is a value function: it gives us the optimal value of profit at different p).
– $\pi(p) \equiv \max_{y \in Y} p \cdot y$ if $\pi(p)$ is achieved.
– $\pi(p) \equiv \sup_{y \in Y} p \cdot y$ if $\pi(p)$ is not achieved.

313 Definition 62 Isoprofitline

Isoprofit line
Isoprofit line: $\overline{\pi} = p_1 y_1 + p_2 y_2 + \ldots + p_n y_n$.

- With 2 goods (2D case), $\overline{\pi} = p_1 y_1 + p_2 y_2$, $y_1 < 0$ and $y_2 > 0$.
  Or $y_2 = \frac{\overline{\pi}}{p_2} - \frac{p_1}{p_2} y_1$.
  It's a line!
- With 3 goods (3D case), we have $y_3 = \frac{\overline{\pi}}{p_3} - \frac{p_1}{p_3} y_1 - \frac{p_2}{p_3} y_2$.
  It's a plane!
- With 4 goods (4D case), we have $y_4 = \frac{\overline{\pi}}{p_4} - \frac{p_1}{p_4} y_1 - \frac{p_2}{p_4} y_2 - \frac{p_3}{p_4} y_3$.
  It's a hyperplane!

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314 4.4. Rationalizability

315 Definition 63 Weak Axiom of Profit Maximization (WAPM)

The Weak Axiom of Profit Maximzation comes from two inferences:

- True Y includes observed production y
- True Y lies below the current isoprofit line py.

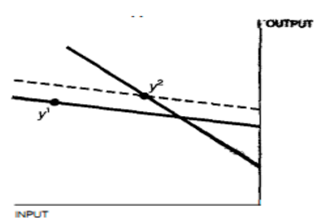
Definition 10.

(WAPM) Supply correspondence $\tilde{y}$ is rationalizable if and only if

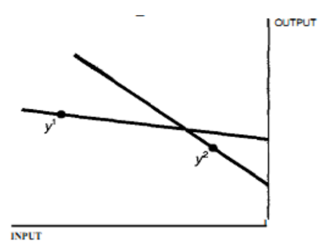
$$p \cdot y' \leq p \cdot y, \quad \forall p, p' \in \mathbb{R}^n, y \in \tilde{y}(p), y' \in \tilde{y}(p')$$

If the firm chose y when facing prices p, must do better than y', which was the choice when facing prices p'.

WAPM is not satisfied:
$p^1 y^2 > p^1 y^1$.



WAPM is satisfied:
$p^1 y^2 < p^1 y^1$.



Theorem (necessary conditions for rationalizability)

Part (i) Any rationalizable profit function π is convex.
Part (ii) Any rationalizable profit function π is homogeneous of degree one, i.e., $\pi(\lambda p) = \lambda \pi(p)$ for all $p \in \mathbb{R}^n$, $\lambda > 0$.
Part (iii) Any optimal supply correspondence Y^* is homogeneous of degree zero, i.e., $Y^*(\lambda p) = Y^*(p)$ for all $p \in \mathbb{R}^n$, all $\lambda > 0$.

Theorem (part (i))

$\pi(\cdot)$ is a convex function.

Proof.

Fix any p_1, p_2 and let $p_t \equiv tp_1 + (1-t)p_2$ for $t \in [0,1]$. Then for any $y \in Y$,

$$\begin{aligned} p_t \cdot y &= t \underbrace{p_1 \cdot y}_{\leq \pi(p_1)} + (1-t) \underbrace{p_2 \cdot y}_{\leq \pi(p_2)} \\ &\leq t\pi(p_1) + (1-t)\pi(p_2). \end{aligned}$$

Since this is true for all $p_t \cdot y$, it holds for $\sup_{y \in Y} p_t \cdot y = \pi(p_t)$:

$$\pi(p_t) \leq t\pi(p_1) + (1-t)\pi(p_2).$$

□

Theorem (part (ii))

$\pi(\cdot)$ is homogeneous of degree one, i.e., $\pi(\lambda p) = \lambda \pi(p)$ for all p and $\lambda > 0$.

That is, if you scale all (input and output) prices up or down the same amount, you also scale profits by that amount.

Proof.

$$\begin{aligned} \pi(\lambda p) &\equiv \sup_{y \in Y} \lambda p \cdot y \\ &= \lambda \sup_{y \in Y} p \cdot y \\ &= \lambda \pi(p). \end{aligned}$$

□

Theorem (part (iii))

$Y^*(\cdot)$ is homogeneous of degree zero; i.e., $Y^*(\lambda p) = Y^*(p)$ for all p and $\lambda > 0$.

That is, a firm makes the same production choice if all (input and output) prices are scaled up or down the same amount

Proof.

$$\begin{aligned} Y^*(\lambda p) &\equiv \{y \in Y \colon \lambda p \cdot y = \pi(\lambda p)\} \\ &= \{y \in Y \colon \lambda p \cdot y = \lambda \pi(p)\} \\ &= \{y \in Y \colon \quad p \cdot y = \quad \pi(p)\} \\ &= Y^*(p). \end{aligned}$$

□

316 **Definition 64 Euler's Law**

Theorem (Euler's Law)

Suppose $f(\cdot)$ is differentiable. Then it is homogeneous of degree k iff $x \cdot \nabla f(x) = kf(x)$.

Corollary

If $f(\cdot)$ is homogeneous of degree one, then $\nabla f(\cdot)$ is homogeneous of degree zero.

Proof.

Homegeneity of degree one means

$$\lambda f(x) = f(\lambda x).$$

Differentiating it in x,

$$\begin{aligned} \lambda \nabla f(x) &= \lambda \nabla f(\lambda x) \\ \nabla f(x) &= \nabla f(\lambda x). \end{aligned}$$

□

317 Definition 65 Hotelling's Lemma

Theorem (Euler's Law)

Suppose $f(\cdot)$ is differentiable. Then it is homogeneous of degree k iff $x \cdot \nabla f(x) = kf(x)$.

Corollary

If $f(\cdot)$ is homogeneous of degree one, then $\nabla f(\cdot)$ is homogeneous of degree zero.

Proof.

Homegeneity of degree one means

$$\lambda f(x) = f(\lambda x).$$

Differentiating it in x,

$$\begin{aligned} \lambda \nabla f(x) &= \lambda \nabla f(\lambda x) \\ \nabla f(x) &= \nabla f(\lambda x). \end{aligned}$$

□

318 Definition 66 Law of Supply

- To obtain a finite-change version of this condition, write a double application of WAPM:

$$\begin{array}{cc} p' y(p') \geq p' y(p) & py(p) \geq py(p') \\ p'(y(p') - y(p)) \geq 0 & -p(y(p') - y(p)) \geq 0 \end{array}$$

  Add to get

$$(p' - p) \cdot (y(p') - y(p)) \geq 0$$

- That is, $\triangle p \triangle y \geq 0$.
- This inequality is known as the "Law of Supply".
- The differentiable version: $dp \cdot dy \geq 0$. That is, supply changes in the direction of the price change.

319 4.5. Cost Minimization Problem (CMP)

320 Definition 67 Single-output Firm

Recall notation for a single-output firms:

$$\begin{aligned} p &\in \mathbb{R}_+\text{: Price of output} \\ w &\in \mathbb{R}_+^{n-1}\text{: Prices of inputs} \\ q &\in \mathbb{R}_+\text{: Output produced} \\ z &\in \mathbb{R}_+^{n-1}\text{: Inputs used} \end{aligned}$$

Thus, a price vector is $(p, w) \in \mathbb{R}^n$ and a net output vector (production plan) is $(q, -z) \in \mathbb{R}^n$.
We will often denote $m \equiv n - 1$.

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Definition (production function)

For a firm with a single output q (and inputs $-z \in \mathbb{R}^m$), the production function $f : \mathbb{R}^m_+ \to \mathbb{R}_+$ is defined as a maximum output that can be produced from the given amount of inputs: $q = f(z)$, where $z \in \mathbb{R}^m_+$.

Examples

Cobb-Douglas production function: $q = f(z_1, z_2) = z_1^{\alpha} z_2^{1-\alpha}$, $\alpha \in (0, 1)$.
CES production function: $q = f(z_1, z_2) = \left(z_1^{\rho} + z_2^{\rho}\right)^{1/\rho}$, $0 \neq \rho < 1$.

With one output, free disposal, and production function $f(\cdot)$, the production set is

$$Y = \{(q, -z) \colon z \in \mathbb{R}^m_+ \text{ and } f(z) \geq q\}.$$

321 Definition 68 Cost Minimization

- If $p > 0$, the firm will never dispose any output (cannot have $q < f(z)$).
- Instead, it will choose $q = f(z)$ to maximize profit:

$$\max_{z \in \mathbb{R}^m_+} \underbrace{pf(z)}_{\text{revenue}} - \underbrace{w \cdot z}_{\text{cost}}$$

- Let us fix the revenue.
- *Observation:* To maximize profit for a fixed revenue, the firm needs to minimize cost.
- Cost minimization can be viewed as an intermediate step in profit maximization.

We separate the profit maximization problem into two parts:

1. Find a cost-minimizing way to produce a given output level q (cost-minimization problem, CMP).
   - *Cost function* (it's a value function – maximum value of the objective function for $\forall q, w$ (parameters)):

     $$CMP: \qquad c(q, w) \equiv \inf_{z \colon f(z) \geq q} \; w \cdot z,$$

     where *CMP* means "cost minimization problem".
   - *Conditional factor (input) demand correspondence* (it's a solution to CMP)

     $$\begin{aligned} Z^*(q, w) &\equiv \arg \min_{z \colon f(z) \geq q} \; w \cdot z \\ &= \{z \colon f(z) \geq q \text{ and } w \cdot z = c(q, w)\}. \end{aligned}$$

     ("conditional" means that q is given).

2. Find an output level that maximizes the difference between revenue and cost, given the optimal cost function.

   $$OOP: \qquad \max_{q \geq 0} \; \{pq - c(q, w)\},$$

   where *OOP* means "optimal output problem".

322 5. General Equilibrium

323 Definition 69 The Walrasian Model

- L goods $\ell \in \mathcal{L} \equiv \{1, \ldots, L\}$.
- I consumers $i \in \mathcal{I} \equiv \{1, \ldots, I\}$.
- Consumers do not have monetary wealth, but rather an endowment of goods which they can trade or consume.
- $e^i \in \mathbb{R}^L_+$ is endowment of consumer i.
- $x^i \in \mathbb{R}^L_+$ is consumption bundle of consumer i.
- Preferences are represented by utility function $u^i \colon \mathbb{R}^L_+ \to \mathbb{R}$.
- Economy $\mathcal{E} = \left((u^i, e^i)^{i \in \mathcal{I}} \right)$.
- Endogenous prices $p \in \mathbb{R}^L_+$. Each consumer takes p as given.

- Each consumer i solves

$$\max_{x^i \in \mathbb{R}^L_+} u^i(x^i)$$
$$\text{s.t. } p \cdot x^i \leq p \cdot e^i,$$

  where, $p \cdot e^i$ is the consumer's wealth (if he sells his endowment).
- Here, logistics does not matter (he can either sell everything or sell the difference of what he needs).
- Equivalently, he solves

$$\max_{x^i \in B^i(p)} u^i(x^i),$$

  where $B^i(p) \equiv \{x^i \in \mathbb{R}^L_+ \, : \, p \cdot x^i \leq p \cdot e^i\}$ is the budget set for i.

324 Definition 70 Walrasian Equilibrium

Definition (Walrasian equilibrium)

A Walrasian equilibrium for economy $\mathcal{E}$ is price p and quantities $(x^i)^{i \in \mathcal{I}}$ such that

1. All agents maximize their utilities; i.e., for all $i \in \mathcal{I}$,

$$x^i \in \arg \max_{x \in B^i(p)} u^i(x);$$

2. Markets clear; i.e., for all $\ell \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x^i_\ell = \sum_{i \in \mathcal{I}} e^i_\ell.$$

325 Definition 71 Pareto Optimality / Pareto Efficiency

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Definition (feasible allocation)

An allocation $(x^i)^{i \in \mathcal{I}} \in \mathbb{R}_+^{I \times L}$ is feasible iff for all goods $\ell \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x_\ell^i \leq \sum_{i \in \mathcal{I}} e_\ell^i.$$

Definition (Pareto optimality)

Given an economy $\mathcal{E}$, a feasible allocation $x \equiv (x^i)^{i \in \mathcal{I}}$ is Pareto optimal if there is no other feasible allocation $\hat{x}$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$ with strict inequality for some i.

326 Definition 72 Edgeworth Box

A1. $u^i(\cdot)$ is continuous;

A2. $u^i(\cdot)$ is increasing; i.e., $x^{i\prime} \gg x^i \Rightarrow u^i(x^{i\prime}) > u^i(x^i)$;

A3. $u^i(\cdot)$ is concave;

A4. $e^i \gg \mathbf{0}$; i.e., every agent has at least a little bit of every good.

Definition (monotone)

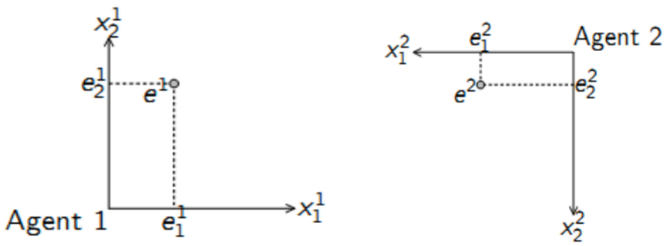
$\succeq$ is monotone iff $x > y \implies x \succ y$.

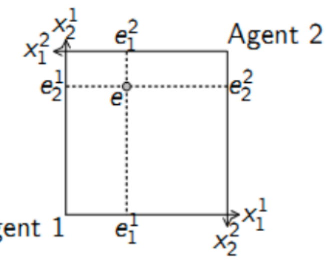
Definition (local non-satiation)

$\succeq$ is locally non-satiated iff for any x and $\epsilon > 0$, there exists y such that $\|x - y\| \leq \epsilon$ and $y \succ x$.

i.e., there are some desirable commodities.

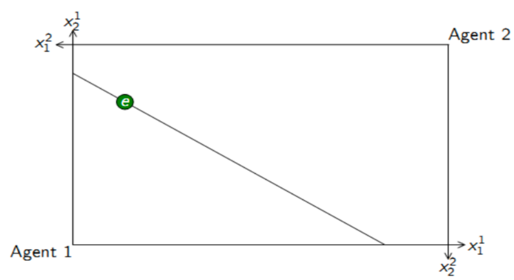
- A2 means (strict) monotonicity. It implies local nonsatiation.
- By A3, if $u^i(\cdot)$ is concave $\Rightarrow$ quasiconcave. Both mean convexity of preferences.
- We do not want to get maximum generality of our results:
  - A2 can be weakened to local nonsatiation.
  - A3 can be weakened to quasiconcavity (preferences are convex).



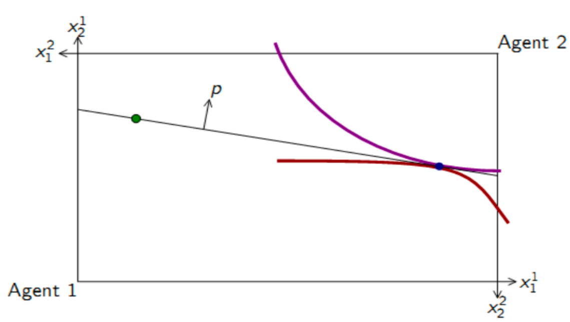


Agents 1 and 2 have the same budget line:

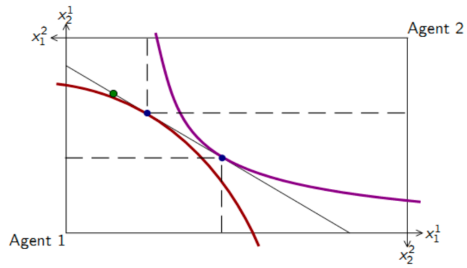
$$p \cdot x^1 = p \cdot e^1 \text{ and } p \cdot x^2 = p \cdot e^2$$



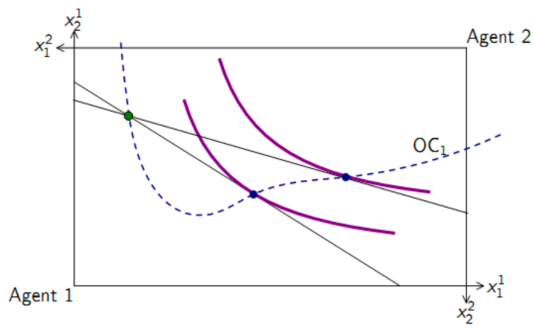
Equilibrium if the agents choose the same point.

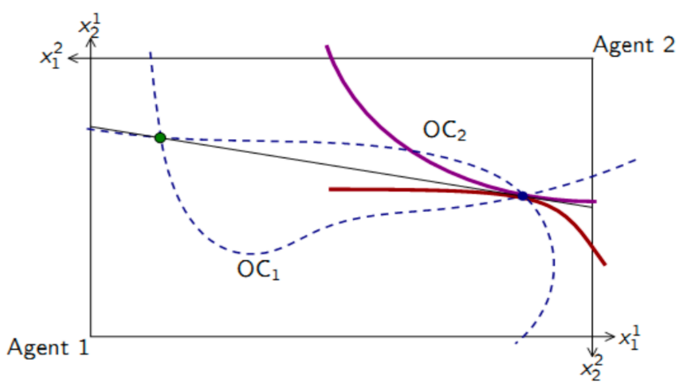


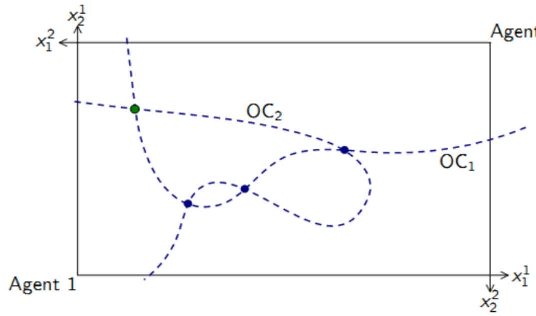
Because the agents choose different points, there is no market clearing at these prices (non-equlibrium prices).



The offer curve traces out Marshallian (Walrasian) demand as prices change. Note that the offer curve starts at e.

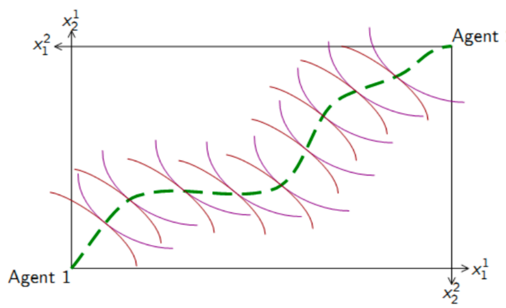




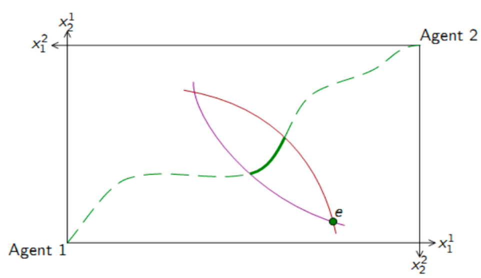


Generically, there is an odd number of equilibria.

The Pareto set is the locus of Pareto optimal allocations.



We expect agents to reach the contract curve: the portion of the Pareto set that makes each better off than e.



If the agents can bargain, they can observe that they can move to the points inside the contract curve.

327 **Definition 73 Walrasian Equilibrium and Pareto Optimality**

WE and PO are very different concepts.

Pareto optimum

1. An allocation

given aggregate endowments and individual preferences.

Walrasian equilibrium

1. An allocation
2. Prices

given individual endowments and preferences.

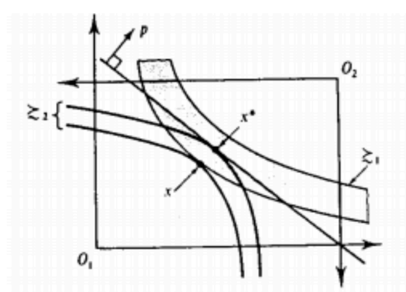
- We want to know the connection between WE and PO. $\Rightarrow$ Welfare Theorems.
- Welfare Theorems impose conditions under which:
  - WE $\Rightarrow$ PO (First Welfare Theorem).
  - PO $\Rightarrow$ WE (Second Welfare Theorem).

328 Definition 74 First Welfare Theorem

Theorem (First Welfare Theorem)

(Arrow, 1951, Debreu, 1951). Let $\left(p, (x^i)^{i\in\mathcal{I}}\right)$ be a Walrasian equilibrium. Then if preferences are locally nonsatiated, the allocation $(x^i)^{i\in\mathcal{I}}$ is Pareto optimal.

A WE x^* is not PO because agent 1's preferences fail to satisfy local nonsatiation (in turn, x is PO).



329 Definition 75 Second Welfare Theorem

Theorem (Second Welfare Theorem)

(Arrow, 1951, Debreu, 1951). Let $\mathcal{E}$ be an economy that satisfies:

A1. *$u^i(\cdot)$ is continuous;*

A2. *$u^i(\cdot)$ is increasing; i.e., $u^i(x^{i\prime}) > u^i(x^i)$ whenever $x^{i\prime} \gg x^i$;*

A3. *$u^i(\cdot)$ is concave;*

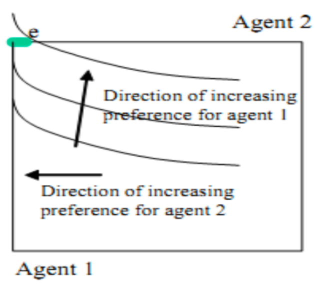
A4. *$e^i \gg \mathbf{0}$; i.e., every agent has at least a little bit of every good.*

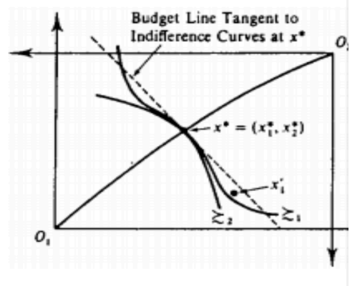
If $(e^i)^{i\in\mathcal{I}}$ is Pareto optimal, then there exists a price vector $p \in \mathbb{R}^L_+$ such that $\left(p, (e^i)^{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium for $\mathcal{E}$.

Examples of failure of the Second Welfare Theorem

Example 1: Arrow's exceptional case ($e_1^1 = 0$ & $e_2^2 = 0$; if $p_2 = 0$, agent 1 has 0 wealth).

Example 2: Nonconvex preferences.





330 Definition 76 Finding a Pareto Optimal Allocation Using Its Definition

Definition (Pareto optimality)

Given an economy $\mathcal{E}$, a feasible allocation $x \equiv (x^i)^{i \in \mathcal{I}}$ is Pareto optimal if there is no other feasible allocation $\hat{x}$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$ with strict inequality for some i.

- Construct an algorithm for finding a PO allocation based on the above definition (later we'll see a different construction).

Step 1. Choose the utility levels attained by agents $i \in \{2, \ldots, I\}$ and fix them.
Step 2. Maximize agent 1's utility subject to the utility levels of the other agents, fixed in Step 1.

- Find a PO allocation by solving the following problem:

$$\max_{(x^i)^{i \in \mathcal{I}} \in \mathbb{R}_+^{I \times L}} u^1(x^1)$$

such that

$$\begin{array}{lll} (1) & u^i(x^i) \geq \bar{u}^i & \text{for } i = 2, \ldots, I \\ (2) & \sum_{i \in \mathcal{I}} x_\ell^i \leq \sum_{i \in \mathcal{I}} e_\ell^i & \text{for } \ell = 1, \ldots, L. \end{array}$$

- We refer to constraints (1) as promise-keeping constraints, and (2) as the economy's resource constraints (or feasibility constraints).

- **A Pareto problem**:

$$\max_{(x^i)^{i\in\mathcal{I}}\in\mathbb{R}_+^{I\times L}} u^1(x^1)$$

such that

$$\begin{array}{lll} (1) & x \geq \mathbf{0} & \\ (2) & u^i(x^i) \geq \bar{u}^i & \text{for } i = 2, \ldots, I \\ (3) & \sum_{i\in\mathcal{I}} x_\ell^i \leq \sum_{i\in\mathcal{I}} e_\ell^i & \text{for } \ell = 1, \ldots, L. \end{array}$$

- By our normalization, $u^i(x^i) > 0$. So, $\bar{u}^i \geq 0$ for all $i \in \{2, \ldots, I\}$.
- Let λ^i = multiplier on $u^i(x^i) \geq \bar{u}^i$.
- Let μ_ℓ = multiplier on $\sum_i x_\ell^i \leq \sum_i e_\ell^i$.
- Under our assumptions, these constraints of type (1) and (2) will be binding at a solution.
- Thus, $\lambda^i > 0$ for $i = 2, \ldots, I$ and $\mu_\ell > 0$ for $\ell = 1, \ldots, L$.

$$\mathcal{L} = u^1(x^1) + \sum_{i=2}^{I} \lambda^i \left(u^i(x^i) - \bar{u}^i\right) + \sum_{\ell=1}^{L}\sum_{i=1}^{I} \left[\mu_\ell(e_\ell^i - x_\ell^i) - \gamma_\ell^i\left(-x_\ell^i\right)\right],$$

where γ_ℓ^i = Lagrange multiplier associated with constraint $x_\ell^i \geq 0$.

- FOC wrt x_ℓ^1 :

$$\frac{\partial u^1}{\partial x_\ell^1} - \mu_\ell + \gamma_\ell^1 = 0.$$

- FOC wrt x_ℓ^i for all $i \in \{2, \ldots, I\}$:

$$\lambda^i \frac{\partial u^i}{\partial x_\ell^i} - \mu_\ell + \gamma_\ell^i = 0.$$

- Thus, the FOC for agent 1 is the same as FOC for all $i \in \{2, \ldots, I\}$ under $\lambda^1 \equiv 1$.
- By complementary slackness, $\gamma_\ell^i x_\ell^i = 0$. 2 possible cases:
  - (i) $\gamma_\ell^i > 0$ and $x_\ell^i = 0$:

$$\lambda^i \frac{\partial u^i}{\partial x_\ell^i} - \mu_\ell \leq 0.$$

  - (ii) $\gamma_\ell^i = 0$ and $x_\ell^i > 0$:

$$\lambda^i \frac{\partial u^i}{\partial x_\ell^i} = \mu_\ell.$$

- Summarizing the FOCs of all agents $i \in \mathcal{I}$:

$$\lambda^i \frac{\partial u^i}{\partial x_\ell^i} \leq \mu_\ell \text{ with equality if } x_\ell^i > 0.$$